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Classical Motions of Spin-$\frac{1}{2}$ Particles*

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From the WKB solutions to the squared Dirac equation we derive the classical trajectories and spin precessions originally postulated by Bargmann, Michel, and Telegdi for charged spin-$\frac{1}{2}$ particles. Identical equations of motion are obtained from the WKB solutions to the Dirac equation.

I. INTRODUCTION

IN 1959, in an analysis based on classical mechanics, Bargmann, Michel, and Telegdi[1] derived the following set of relativistic equations describing the classical trajectories and spin motions of spin-$\frac{1}{2}$ particles in uniform and constant electric and magnetic fields[2]:

$$m(dv_\mu/d\tau)=-(e/c)F_{\mu\nu}v_\nu \tag{1a}$$

$$\frac{dS_\mu}{d\tau}=-(1+g_1)\left(\frac{e}{mc}\right)F_{\mu\nu}S_\nu+\left(\frac{g_1e}{mc^3}\right)F_{\rho\sigma}S_\rho v_\sigma v_\mu. \tag{1b}$$

Their equations (hereafter called the classical equations) predicted particle trajectories and spin precessions which agreed with those obtained in special cases by other authors whose calculations were based on the Dirac theory. These calculations were completely confirmed by experiments performed with polarized beams.[3] The Bargmann, Michel, Telegdi note of 1959 stated so clearly and succinctly the classical limits of the Dirac theory, that soon thereafter attempts were made to derive the classical equations directly from the Dirac equation in the usual limit as h vanishes.

In a recent article, Rubinow and Keller[4] proved that the classical equations do indeed follow from the Dirac equation in the same sense that Newton's equations follow from the Schrödinger equation. Rubinow and Keller found the WKB solutions to the Dirac equation and showed that the classical equations coud be extracted from these asymptotic solutions.

In this note we wish to show how the classical equations may be obtained by another method based on some early work of V. Fock.[5] This method, which is representation invariant, leads to a considerable simplification in the calculations of Rubinow and Keller, shows clearly the relation between the asymptotic solutions of the Dirac equation and the squared (second-order) Dirac equation, and permits one to treat more readily the related problems of the classical motion of charged, spinning particles and the asymptotic solutions to the Dirac equation, in the presence of inhomogeneous fields.[6]

II. WKB SOLUTIONS TO THE DIRAC EQUATION

The Dirac equation, including the Pauli anomalous magnetic moment term, may be written[7]

$$[\pi_\mu\gamma_\mu-imc-(g_1e\hbar/4mc^2)\gamma_\mu\gamma_\nu F_{\mu\nu}]\Psi=0\,, \tag{2}$$

with $\pi_\mu=-i\hbar\partial_\mu+(e/c)A_\mu$. We introduce the wave function Φ,

$$[\pi_\rho\gamma_\rho+imc+(g_1e\hbar/4mc^2)\gamma_\rho\gamma_\sigma F_{\rho\sigma}]\Phi=2mc\Psi \tag{3}$$

and apply the operator $\pi_\mu\gamma_\mu-imc-(g_1e\hbar/4mc^2)\gamma_\mu\gamma_\nu F_{\mu\nu}$ to both sides of Eq. (3). We find that Φ satisfies the equation

$$[\pi_\mu\pi_\mu+m^2c^2+(1+g_1)(e\hbar/2ic)\gamma_\mu\gamma_\nu F_{\mu\nu}+(g_1e\hbar/mc^2)F_{\mu\nu}\gamma_\mu\pi_\nu]\Phi=0\,, \tag{4}$$

when terms depending on $\hbar^2$ are neglected.

We now follow Fock[5] and seek the WKB solutions to the squared Dirac equation, (4). These asymptotic solutions are written as a series in the small parameter $\hbar$,

$$\Phi_{\mathrm{WKB}}=e^{iS/\hbar}\sum_{n=0}^{\infty}(-i\hbar)^n a_n\,, \tag{5}$$

where S is a scalar function and the a_n are four-component spinors. In our present calculation we are interested in only the first term in the expansion. We insert $\Phi_{\mathrm{WKB}}=a_0e^{iS/\hbar}$ into (4), once again neglect terms depending on $\hbar^2$, and require the coefficients of $\hbar^0$ and $\hbar^1$ to vanish separately. We find that S and a_0 satisfy the following equations:

$$(\partial_\mu S+(e/c)A_\mu)^2+m^2c^2=0 \tag{6}$$

and

$$2imv_\mu\partial_\mu a_0+im(\partial_\mu v_\mu)a_0=(1+g_1)(e/2ic)\gamma_\rho\gamma_\sigma F_{\rho\sigma}a_0+(g_1e/c^2)F_{\alpha\beta}\gamma_\alpha\gamma_\beta a_0\,, \tag{7}$$

where $mv_\mu=\partial_\mu S+(e/c)A_\mu$.

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[1] V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Letters **2**, 435 (1959).

[2] In our notation the four velocity is given by v_μ, the spin pseudovector is S_μ, $v_\mu^2=-c^2$, $S_\mu^2=$const., $S_\mu v_\mu=0$, $v_\mu=dx_\mu/d\tau$, $x_\mu\rightarrow(\mathbf{x},ict)$, $F_{ik}=B_l$, $F_{k4}=iE_k$ and τ is the proper time. The anomalous magnetic moment is described by g_1. Our $g_1+1=g/2$ of Bargmann, Michel, and Telegdi, and the sign of our charge e is the negative of theirs.

[3] References to theory and experiment are to be found in Ref. 1.

[4] S. I. Rubinow and J. B. Keller, Phys. Rev. **131**, 2789 (1963).

[5] V. Fock, Physik. Z. Sowjetunion **12**, 404 (1937).

[6] In this note we treat only the motion of particles in homogeneous fields.

[7] We follow the notation in H. S. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Academic Press Inc., New York, 1957), Sec. 10γ.

The WKB solution of the squared Dirac equation is found by solving Eqs. (6) and (7). Equation (6) is the Hamilton-Jacobi equation associated with the classical equation of motion [the classical equation (1a)],

$$m(dv_\mu/d\tau)=-(e/c)F_{\mu\nu}v_\nu. \qquad (1a)$$

We now show (following Fock) that (7) may also be written as an ordinary differential equation with the proper time τ acting as independent variable. We write $a_0=R\phi$, where ϕ is a unit spinor and R a scalar whose square satisfies an equation of continuity,

$$(1/m)\partial_\mu[R^2(\partial_\mu S+(e/c)A_\mu)]\equiv\partial_\mu(R^2v_\mu)=0. \qquad (8)$$

It is not difficult to prove that if we have a solution of the relativistic Hamilton-Jacobi equation, (6), then a solution of (8) is always given by

$$R^2=ic\left\|\frac{\partial^2 S}{\partial x^i\partial\alpha^k}\right\|\left(\frac{dx_4}{d\tau}\right)^{-1}. \qquad (9)$$

The α^k in the determinant (Van Vleck determinant), $\|\partial^2S/\partial x^i\partial\alpha^k\|$, are the three separation constants appearing in the complete solution of the relativistic Hamilton-Jacobi equation.[8]

We return to (7), replace a_0 by $R\phi$, and find that when R^2 satisfies (8), $v_\mu\partial_\mu\phi\equiv d\phi/d\tau$ is given by

$$\frac{d\phi}{d\tau}=-(1+g_1)\left(\frac{e}{4mc}\right)\gamma_\rho\gamma_\sigma F_{\rho\sigma}\phi-\left(\frac{ig_1e}{2mc^2}\right)F_{\rho\sigma}v_\rho\gamma_\sigma\phi. \qquad (10a)$$

Similarly for $\bar\phi=\phi^+\gamma_4$ we have

$$\frac{d\bar\phi}{dt}=(1+g_1)\left(\frac{e}{4mc}\right)\bar\phi\gamma_\rho\gamma_\sigma F_{\rho\sigma}-\left(\frac{ig_1e}{2mc^2}\right)\bar\phi F_{\rho\sigma}v_\rho\gamma_\sigma. \qquad (10b)$$

For the case of uniform constant fields, (10a) and (10b) are ordinary differential equations, since the velocity vector $v_\mu(\tau)$ is a solution of (1a) and therefore a known function of τ.

We now have Eq. (1a) and Eqs. (10) as the classical equations of motion (both are independent of $\hbar$) associated with the squared Dirac equation. Equation (1a) yields the trajectory of the particle, while the unit spinor ϕ of (10) in some way describes its spin (intrinsic rotation). Before we find the classical spin equation, we return to analyze the WKB solution of the squared Dirac equation,

$$\Phi_{\mathrm{WKB}}=R\phi e^{iS/\hbar}. \qquad (11)$$

All along we have insisted that ϕ is a unit spinor, i.e., $\bar\phi\phi=1$. This requirement has been made so that the probability interpretation of the wave function Φ could be maintained even in the asymptotic limit of the wave theory. For if $(\bar\Phi\Phi)_{\mathrm{WKB}}$ is to represent the relative numbers of particles in a given (four) volume element, then in the classical limit $(\bar\Phi\Phi)_{\mathrm{WKB}}$ must satisfy the equation of continuity

$$\partial_\mu[v_\mu(\bar\Phi\Phi)_{\mathrm{WKB}}]=0. \qquad (12)$$

Since R^2 satisfies this equation, ϕ must be a unit spinor. However, from Eqs. (10) we find

$$d(\bar\phi\phi)/d\tau=-(ig_1e/mc^2)F_{\alpha\beta}\bar\phi\gamma_\alpha\phi v_\beta, \qquad (13)$$

so that ϕ is a unit spinor only if $\bar\phi\gamma_\alpha\phi\propto v_\alpha$; otherwise the right hand side of Eq. (13) fails to vanish. Since v_α is a time-like vector ($v_\alpha v_\alpha=-c^2$), we require that

$$ic\bar\phi\gamma_\alpha\phi=v_\alpha, \quad \bar\phi\phi=1. \qquad (14)$$

The above identification is consistent, for with it $ic\bar\phi\gamma_\alpha\phi$ may be shown to satisfy the same equation as v_α,

$$m(d/d\tau)(ic\bar\phi\gamma_\alpha\phi)=-(e/c)F_{\alpha\beta}(ic\bar\phi\gamma_\alpha\phi). \qquad (15)$$

In addition to the requirement $\bar\phi\phi=1$, we may also fix $\bar\phi\gamma_5\phi=0$, since if we choose this value initially, it is maintained for all time by the equation of motion $(d/d\tau)(\bar\phi\gamma_5\phi)=0$.

Once the classical velocity vector has been equated to the bilinear form $ic\bar\phi\gamma_\mu\phi$, it seems reasonable to attempt to relate the classical spin pseudovector S_μ to the bilinear form $\bar\phi\gamma_5\gamma_\mu\phi$. Indeed, if, in analogy with (14), we write

$$S_\mu=i\hbar\bar\phi\gamma_5\gamma_\mu\phi, \qquad (16)$$

we then find the following dynamical equation for S_μ:

$$\frac{dS_\mu}{d\tau}=-(1+g_1)\left(\frac{e}{mc}\right)F_{\mu\nu}S_\nu+\left(\frac{g_1e}{mc^3}\right)F_{\alpha\beta}S_\alpha v_\beta v_\mu. \qquad (1b)$$

In the derivation of the last equation, (1b), which is the spin equation of Bargmann, Michel, and Telegdi, we have made use of the following identity:

$$\tfrac{1}{2}\bar\phi(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\gamma_5\phi=(c\hbar)^{-1}(S_\mu v_\nu-S_\nu v_\mu), \qquad (17)$$

where v_μ and S_μ are given by Eqs. (14) and (16), and $\bar\phi\phi=1$, $\bar\phi\gamma_5\phi=0$.

It should be emphasized that the Dirac electron theory predicts, and experiment confirms, that, in the classical limit, spin-$\frac{1}{2}$ particles have both space-time trajectories and intrinsic moments (spin). It should also be noted that the classical equations, (1), set A, may be replaced by (1a), (10a), and the definitions (14) and (16), set B. The latter choice (set B) corresponds to the use of spinors in, for example, the classical theory of the top.[9] However, in a classical theory, physical meaning can be attributed only to the expectation values (bilinear forms) associated with the spinor, so that both sets of classical equations, A and B, are equivalent. Nevertheless, if we recognize that the relativistic quantum theory predicts quantization of spin and, under usual circumstances, a gap between positive

[8] R. Schiller, Phys. Rev. **125**, 1100 (1962).

[9] F. Klein, *The Mathematical Theory of the Top* (Scribners and Sons, New York, 1897).

and negative energy states, then the solution for ϕ in (10a) indicates the relative numbers of particles which have been transformed by the interaction from a given spin and energy state at some initial time, into other states of spin and energy at some later time. In this sense, Eqs. (10) for the spinor ϕ yield more information, at least from an interpretive point of view, than Eqs. (1) for the bilinear forms v_μ and S_μ.

The classical equations may be determined in exactly the same manner from the asymptotic solutions of the Dirac equation (instead of the squared Dirac equation); for once we have found an asymptotic solution, $\Phi_{\rm WKB}$, in the form of Eq. (11), Eq. (3) then gives us the appropriate WKB approximation to the Dirac equation:

$$2mc\Psi_{\rm WKB}=m(v_\mu\gamma_\mu+ic)\Phi_{\rm WKB}. \tag{18}$$

If we write

$$2mc\Psi_{\rm WKB}=R\psi e^{iS/\hbar}, \tag{19}$$

ψ is easily shown to satisfy the relation $\bar{\psi}\psi=$const, and the bilinear forms associated with ψ satisfy the following relations:

$$ic\bar{\psi}\gamma_\mu\psi=v_\mu, \tag{20a}$$

$$i\hbar\bar{\psi}\gamma_5\gamma_\mu\psi=S_\mu. \tag{20b}$$

In deriving (20), v_μ and S_μ are defined by Eqs. (14) and (16), and the identity (17) is invoked. If we interpret the left-hand sides of Eqs. (20) as the velocity and spin of a classical particle, we arrive at the classical equations.

III. CONCLUSION

In this note we have shown that the classical equations of Bargmann, Michel, and Telegdi may be derived from either the asymptotic solutions to the Dirac equation or the squared Dirac equation. In a future paper we shall discuss quantization of these WKB solutions, as well as the many analogies existing between the classical theory and the quantum theory of spinning particles.

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Classically Radiationless Motions and Possible Implications for Quantum Theory

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A simple general criterion is developed, using the retarded potentials of classical electromagnetic theory, for absence of radiation from arbitrary time-periodic charge-current distributions. The criterion is applied to rigid finitely extended distributions of charge which may undergo orbital motion with period T. It is found that, for this type of distribution, the condition for no radiation is that the extent b of the distribution be an integer multiple of cT. Some of these distributions may spin while orbiting. There exists at least one asymmetric spinning distribution which doesn't radiate under this condition; for this distribution, the (constant) spin angular velocity must be proportional to an integer ≥ 0 times c/b. This leads to the result that that part of the total (electromagnetic) angular momentum which is associated with the spin angular velocity must be an integer ≥ 0 times e^2/c times a numerical constant whose value depends on the details of the distribution. It is shown that, when such nonradiating distributions are considered as stable particles, there exists an intrinsic uncertainty relation of the same form and with almost the same meaning as that of quantum theory.

I. INTRODUCTION

It still seems a fairly common belief that there exist no nontrivial charge-current distributions which do not radiate, according to classical electromagnetic theory retarded potential solutions. However, early in this century Sommerfeld,[1] Herglotz,[2] and Hertz[3] considered extended electron models, and established the existence of radiationless self-oscillations. In 1933, Schott[4] showed that a uniformly charged spherical shell will not radiate while in orbital motion with period T, provided the shell radius is an integral multiple of $cT/2$; the orbit need not be circular nor even planar. In 1948, Bohm and Weinstein[5] found several other rigid spherically symmetric distributions which can oscillate linearly without radiating.

In this paper we derive a simple exact criterion for absence of radiation, and apply it to moving rigid extended charge distributions.[6] We find that there are many such distributions, some of which may "spin," and others which need not be spherically symmetric.

[1] A. Sommerfeld, Nachr. Akad. Wiss. Goettingen, Math.-Physik. Kl. IIa Math.-Physik. Chem. Abt. **1904**, 99 and 363; **1905**, 201.
[2] G. Herglotz, Nachr. Akad. Wiss. Goettingen, Math.-Physik. Kl. IIa Math.-Physik. Chem. Abt. **1903**, 357; Math. Ann. **65**, 87 (1908).
[3] P. Hertz, Math. Ann. **65**, 1 (1908).
[4] G. A. Schott, Phil. Mag. Suppl. 7, **15**, 752 (1933).
[5] D. Bohm and M. Weinstein, Phys. Rev. **74**, 1789 (1948).
[6] In Bull. Am. Phys. Soc. **9**, 148 (1964), which I received while writing this paper, there appears an abstract by S. M. Prastein and T. Erber which implies that some of the content of this paper has been worked out independently by these authors.